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Accurate numerical solution and analytical approximation for the wind profile over flat terrain

Genaue numerische Lösung und analytische Näherung für das Windprofil über ebenem Gelände

Ulf Janicke, Lutz Janicke

April 2017

Ingenieurbüro Janicke, Hermann-Hoch-Weg 1, 88662 Überlingen, Germany

Autoren/*Authors* :

Dr. Ulf Janicke
Ing.-Büro Janicke
Hermann-Hoch-Weg 1
D-88662 Überlingen
(Email uj@janicke.de)

Dr. Lutz Janicke
Ing.-Büro Janicke
Hermann-Hoch-Weg 1
D-88662 Überlingen
(Email lj@janicke.de)

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Accurate numerical solution and analytical approximation for the wind profile over flat terrain

Ulf Janicke, Lutz Janicke

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Abstract

An accurate numerical procedure for calculating the vertical profile of wind speed and wind direction over flat terrain for a given exchange coefficient profile is presented together with an analytical approximation. The profiles can be used in local wind or air quality studies over flat terrain. In contrast to simpler analytical profiles, the proposed profiles provide a consistent treatment of both wind speed and wind direction, as well close to the ground. In contrast to prognostic wind field models, the proposed methods demand considerably less computation power. However, they provide a smooth transition to the profiles resulting from prognostic wind field models in the limit of flat and homogeneous terrain.

Zusammenfassung

Es wird ein genaues numerisches Lösungsverfahren für die Vertikalprofile von Windgeschwindigkeit und Windrichtung über ebenem Gelände für ein vorgegebenes Profil des Austauschkoeffizienten angegeben sowie eine analytische Näherungslösung. Die Profile können für lokale Wind- und Ausbreitungsuntersuchungen über ebenem Gelände verwendet werden. Im Gegensatz zu einfacheren analytischen Ansätzen erlauben sie eine konsistente Beschreibung von Windgeschwindigkeit und Windrichtung, letztere insbesondere in Bodennähe. Im Gegensatz zu prognostischen Windfeldmodellen benötigen sie wesentlich weniger Rechnerleistung, erlauben aber dennoch einen relativ glatten Übergang zu den Ergebnissen solcher Modelle im Grenzfall ebenen und homogenen Geländes.

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1 Introduction

The motion of a volume element in the atmosphere in the absense of dissipation is governed by the Euler equation

$$\rho \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v} = -\mathrm{grad}p \tag{1}$$

where ρ is the density of air, $\boldsymbol{v}$ the velocity vector, and p the air pressure.

An earth-fixed coordinate system is assumed with the z-axis perpendicular to the ground at geographic latitude φ, the x-axis pointing from west to east and the y-axis from south to north. The equations of motion for the horizontal vector $\boldsymbol{v} = (u, v)$ read (Ω is the earth rotation frequency):[1]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - 2\Omega(w\cos\varphi - v\sin\varphi) \tag{2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\varphi \tag{3}$$

For stationary, homogeneous flow conditions this reduces to

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + f_\mathrm{c} v \tag{4}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} - f_\mathrm{c} u \tag{5}$$

with the Coriolis frequency $f_\mathrm{c} = 2\Omega\sin\varphi$ ($f_\mathrm{c} \approx 10^{-4}$ 1/s at Central European latitude). These equations define the geostrophic wind vector $\boldsymbol{g} = (u_\mathrm{g}, v_\mathrm{g})$ with the horizontal components

$$u_\mathrm{g} = -\frac{1}{\rho f_\mathrm{c}}\frac{\partial p}{\partial y} \tag{6}$$

$$v_\mathrm{g} = \frac{1}{\rho f_\mathrm{c}}\frac{\partial p}{\partial x} \tag{7}$$

In reality there is dissipation due to frictional forces at the ground, which can be accounted for in the equations of motion in form of a dissipation term parameterized by an exchange coefficient for momentum, K_m:

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + f_\mathrm{c} v + \frac{\partial}{\partial z}\left(K_\mathrm{m}\frac{\partial u}{\partial z}\right) \tag{8}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} - f_\mathrm{c} u + \frac{\partial}{\partial z}\left(K_\mathrm{m}\frac{\partial v}{\partial z}\right) \tag{9}$$

[1]See for example SEINFELD, J.H., PANDIS, S.N.: *Atmospheric Chemistry and Physics*. Wiley & Sons, New York, 2006. Effects of the centrifugal force are neglected, or, in other words, it is assumed that the z-axis is perpendicular to the geopotential surface.

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Replacing the pressure gradients by the components of the geostrophic wind yields

$$0 = -f_c(v_g - v) + \frac{\partial}{\partial z}\left(K_m \frac{\partial u}{\partial z}\right) \tag{10}$$

$$0 = f_c(u_g - u) + \frac{\partial}{\partial z}\left(K_m \frac{\partial v}{\partial z}\right) \tag{11}$$

If the Coriolis force is neglected ($f_c \to 0$) and the coordinate system is aligned such that $v = 0$, the equations of motions take the simple form

$$0 = \frac{\partial}{\partial z}\left(K_m \frac{\partial u}{\partial z}\right) \quad \longrightarrow \quad K_m \frac{\partial u}{\partial z} = \text{const.} \tag{12}$$

Then the assumption $K_m \propto z$ immediately yields the well known logarithmic wind profile.

In general, however, the Coriolis force is not negligible and the vertical profile of K_m is more complex. The following sections present for the general case an efficient, accurate solution of the equations of motion (10) and (11) and a closed analytical approximation.[2]

2 Numerical solution

Starting point for the numerical solution are the equations of motion (10) and (11), here rewritten in the form

$$\frac{\partial}{\partial z}\left(K_m \frac{\partial u}{\partial z}\right) = -f_c(v - v_g) \tag{13}$$

$$\frac{\partial}{\partial z}\left(K_m \frac{\partial v}{\partial z}\right) = f_c(u - u_g) \tag{14}$$

2.1 Derivation

The boundary conditions for the wind vector $\boldsymbol{v}(z)$ are

$$\boldsymbol{v}(z \to \infty) = \boldsymbol{g} \tag{15}$$

$$\boldsymbol{v}(0) = 0 \tag{16}$$

[2]This work was presented at the 16th EMS Annual Meeting, Triest: Janicke & Janicke (2016), *Accurate numerical solution and analytical approximation for the wind profile over flat terrain.*

It is useful to write the equations of motion in dimensionless form using the normalization constants

$$K_{\mathrm{ref}} = \text{reference value, e.g. maximum of } K_{\mathrm{m}}(z) \tag{17}$$
$$l_{\mathrm{ref}} = \sqrt{K_{\mathrm{ref}}/f_{\mathrm{c}}} \tag{18}$$
$$v_{\mathrm{ref}} = |\boldsymbol{g}| \tag{19}$$

The normalized variables read

$$\tilde{z} = z/l_{\mathrm{ref}} \tag{20}$$
$$k = K_{\mathrm{m}}/K_{\mathrm{ref}} \tag{21}$$
$$\tilde{\boldsymbol{v}} = \boldsymbol{v}/|\boldsymbol{g}| \tag{22}$$
$$\tilde{\boldsymbol{g}} = \mathbf{1} \tag{23}$$

and the equations of motion become (the apostrophe denotes the partial derivative with respect to $\tilde{z}$)

$$(k\tilde{u}')' = -(\tilde{v} - \tilde{v}_{\mathrm{g}}) \tag{24}$$
$$(k\tilde{v}')' = \tilde{u} - \tilde{u}_{\mathrm{g}} \tag{25}$$

Without loss of generality the coordinate system can be chosen such that the x-axis is aligned with the geostrophic wind and hence $\tilde{u}_{\mathrm{g}} = 1$ and $\tilde{v}_{\mathrm{g}} = 0$. Then, using the complex variable

$$\mu = \tilde{u} + i\tilde{v} \tag{26}$$

the equations of motion (24) and (25) can be written in combined form simply as

$$(k\mu')' = i(\mu - 1) \tag{27}$$

with the boundary conditions

$$\mu(\tilde{z} \to \infty) = 1 \tag{28}$$
$$\mu(0) = 0 \tag{29}$$

2.2 Simple case: constant exchange coefficient

For the simple case of a vertically constant exchange coefficient $k(\tilde{z}) = k_0$, Eq. (27) reduces to the oscillator equation

$$\omega^{-2}\mu'' - \mu = -1 \tag{30}$$
$$\omega^2 = i/k_0 \tag{31}$$

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Two linearly independent solutions of the homogeneous differential equation are $\mu_1(\tilde{z}) = \exp(\omega\tilde{z})$ and $\mu_2(\tilde{z}) = \exp(-\omega\tilde{z})$. A special solution of the inhomogeneous equation is $\mu_0(\tilde{z}) = 1$. Hence the general solution is of the form

$$\mu(\tilde{z}) = 1 + a\exp(\omega\tilde{z}) + b\exp(-\omega\tilde{z}) \quad (32)$$

with some factors a and b. To determine these factors, the exponential function is separated into real and imaginary part:

$$\omega = \lambda + i\lambda \quad (33)$$
$$\lambda = 1/\sqrt{2k_0} \quad (34)$$
$$\exp(\omega\tilde{z}) = \exp(\lambda\tilde{z})\ [\cos(\lambda\tilde{z}) + i\sin(\lambda\tilde{z})] \quad (35)$$
$$\exp(-\omega\tilde{z}) = \exp(-\lambda\tilde{z})\ [\cos(\lambda\tilde{z}) - i\sin(\lambda\tilde{z})] \quad (36)$$

The boundary condition (28) requires $a = 0$, the boundary condition (29) requires $b = -1$. The simple case $k(\tilde{z}) = k_0$ thus yields the Ekman spiral

$$\mu(\tilde{z}) = 1 - \exp(-\lambda\tilde{z})\ [\cos(\lambda\tilde{z}) - i\sin(\lambda\tilde{z})] \quad (37)$$

2.3 General case: arbitrary exchange coefficient

In order to solve Eq. (27) in an efficient and accurate way it is assumed that $k(\tilde{z})$ is constant or is set constant above a given height $\hat{z}$:

$$k(\tilde{z} > \hat{z}) = \hat{k} \quad (38)$$

This is a decent assumption as the exchange coefficient decreases towards the mixing layer height to very small values that can be approximated by some small value $\hat{k}$.

Then, for $\tilde{z} > \hat{z}$ the solution $\mu(\tilde{z})$ must have the form of the Ekman spiral derived in the preceding section

$$\hat{\mu}(\tilde{z} > \hat{z}) = 1 + b\exp(-\omega\tilde{z}) \quad (39)$$

with some constant b.

For $\tilde{z} \leq \hat{z}$ the solution $\mu(\tilde{z})$ of the differential equation (27) must connect continuously and with a continuous first derivative to $\hat{\mu}(\tilde{z})$ at $\tilde{z} = \hat{z}$:

$$\mu(\hat{z}) = 1 + b\exp(-\omega\hat{z}) \quad (40)$$
$$\mu'(\hat{z}) = -b\omega\exp(-\omega\hat{z}) \quad (41)$$

The constant b can be eliminated from these two equations and the boundary condition for a solution $\mu(\tilde{z})$ that connects to the Ekman spiral reads

$$\mu(\hat{z}) = 1 - \frac{\mu'(\hat{z})}{\omega} \quad (42)$$

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The general solution of the inhomogeneous differential equation (27) is again the sum of a special solution $\mu_0(\tilde{z})$ and two linearly independent solutions $\mu_1(\tilde{z})$ and $\mu_2(\tilde{z})$ of the homogeneous differential equation:

$$\mu(\tilde{z}) = \mu_0(\tilde{z}) + a_1\mu_1(\tilde{z}) + a_2\mu_2(\tilde{z}) \tag{43}$$

Like before, a special solution μ_0 is a constant,

$$\mu_0(\tilde{z}) = 1 \tag{44}$$

The functions μ_1 and μ_2 are solutions of the differential equation

$$\left[k(\tilde{z})\mu_j'(\tilde{z})\right]' = i\mu_j(\tilde{z}) \tag{45}$$

with $j = 1, 2$. The solutions can be determined numerically, applying for $\mu_1(\tilde{z})$ boundary conditions at the bottom $\tilde{z} = 0$ (integration from bottom to top)

$$\mu_1(0) = 0 \tag{46}$$
$$\mu_1'(0) = 1 \tag{47}$$

and for $\mu_2(\tilde{z})$ boundary conditions at the top $\tilde{z} = \hat{z}$ (integration from top to bottom)

$$\mu_2(\hat{z}) = 0 \tag{48}$$
$$\mu_2'(\hat{z}) = 1 \tag{49}$$

Inserting ansatz (43) into the boundary conditions (29) and (42) yields equations for the constants a_1 and a_2:

$$1 + a_1\mu_1(0) + a_2\mu_2(0) = 0 \tag{50}$$
$$1 + a_1\mu_1(\hat{z}) + a_2\mu_2(\hat{z}) = 1 - \frac{a_1\mu_1'(\hat{z}) + a_2\mu_2'(\hat{z})}{\omega} \tag{51}$$

The solutions are

$$a_2 = \frac{-1}{\mu_2(0)} \tag{52}$$
$$a_1 = \frac{-a_2}{\omega\mu_1(\hat{z}) + \mu_1'(\hat{z})} \tag{53}$$

This determines the normalized, complex wind vector $\mu(\tilde{z})$ in the range $0 \le \tilde{z} \le \hat{z}$. For $\tilde{z} > \hat{z}$ the vector is extended by $\hat{\mu}(\tilde{z})$ of Eq. (39) with the constant

$$b = [\mu(\hat{z}) - 1]\exp(\omega\hat{z}) \tag{54}$$

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2.4 Numerical integration

Equation (45) can be numerically integrated using the Runge-Kutta method. It is straightforward to apply this method to complex functions. However, the differential equation of second order must be transformed into a set of two differential equations of first order. With the additional dependent variables $s_j(\tilde{z}) = \mu_j'(\tilde{z})$ with $j = 1, 2$, the set of homogeneous differential equations reads

$$\mu_j'(\tilde{z}) = s_j(\tilde{z}) \quad (55)$$

$$s_j'(\tilde{z}) = \frac{i\mu_j(\tilde{z}) - k'(\tilde{z})s_j(\tilde{z})}{k(\tilde{z})} \quad (56)$$

with $j = 1, 2$ and the boundary conditions

$$\mu_1(0) = 0 \quad (57)$$

$$s_1(0) = 1 \quad (58)$$

$$\mu_2(\hat{z}) = 0 \quad (59)$$

$$s_2(\hat{z}) = 1 \quad (60)$$

Once μ_0, μ_1, μ_2, a_1, a_2, and thus μ have been determined by numerical integration, the normalized solution can be transformed to the profiles of the velocity components $u(z)$ and $v(z)$ by re-scaling the vertical coordinate to $z = l_{\text{ref}}\tilde{z}$ and by multiplication of μ with the normalization velocity $v_{\text{ref}} = |\boldsymbol{g}|$. The latter can be set in two ways:

1. In the context of prognostic wind field models, the value of the geostrophic wind velocity is usually specified directly as boundary condition.

2. In applications based on wind speed measurements at the ground, the geostrophic wind velocity is not available. Then the relation

$$K_{\text{m}}\frac{\partial u}{\partial z} = u_*^2 \quad (61)$$

which is valid close to the ground, can be used. Together with

$$\frac{\partial u(0)}{\partial z} = |\mu'(0)|\frac{v_{\text{ref}}}{l_{\text{ref}}} \quad (62)$$

it yields the following expression for the normalization velocity:

$$v_{\text{ref}} = |\boldsymbol{g}| = \frac{u_*^2 l_{\text{ref}}}{|\mu'(0)|K_{\text{m}}(0)} \quad (63)$$

In the first case, the geostrophic wind velocity is provided as boundary condition and determines the velocity gradient close to the ground. In the second case, the velocity gradient close to the ground is provided as boundary condition and determines the geostrophic wind velocity.

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2.5 Profile of the exchange coefficient

The numerical integration requires a specification for the vertical profile of the exchange coefficient K_m. For the purpose of this paper the following profile is applied, which is based on common literature and practical applications and which in similar form is applied in the German guideline VDI 3783 Part 8 (2016):

$$K_\mathrm{m}(z) = \kappa u_*(z+z_0) \begin{cases} \dfrac{1}{1+5(z+z_0)/L} e^{-6\alpha z/h_\mathrm{m}} & , \; 1/L \geq 0 \\ \left[e^{-24\alpha z/h_\mathrm{m}} + 15\left(\dfrac{-(z+z_0)}{L}\right)\left(1-0.8\dfrac{z}{h_\mathrm{m}}\right)^8 \right]^{1/4} & , \; 1/L < 0 \end{cases} \quad (64)$$

with the von-Kármán constant $\kappa = 0.4$ and the input parameters friction velocity u_*, surface roughness length z_0, Obukhov length L, mixing layer height h_m, and $\alpha = 0.3$.

This profile does not meet the constraint of being constant above a given height. However, a height $\hat{z}$ can be defined above which K_m is assumed to be constant. For example, if (using again normalized parameters) k takes its maximum value k_max at $\tilde{z} = \tilde{z}_\mathrm{max}$, then $\hat{z}$ is set to the height where $k(z)$ has dropped to the value $f_\mathrm{k} k_\mathrm{max}$,

$$k(\hat{z}) = f_\mathrm{k} k_\mathrm{max} \text{ with the constraint } \hat{z} > \tilde{z}_\mathrm{max} \quad (65)$$

$$\hat{k} = k(\hat{z}) \quad (66)$$

A reasonable choice is for example $f_\mathrm{k} = 0.02$.

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3 Analyical approximation

In extension to the accurate numerical integration presented in the preceding section, this section provides an approximate analytical solution of the equations of motion

$$\frac{\partial}{\partial z}\left(K_\mathrm{m}\frac{\partial u}{\partial z}\right) = -f_\mathrm{c}(v - v_\mathrm{g}) \tag{67}$$

$$\frac{\partial}{\partial z}\left(K_\mathrm{m}\frac{\partial v}{\partial z}\right) = f_\mathrm{c}(u - u_\mathrm{g}) \tag{68}$$

3.1 Derivation

For a constant exchange coefficient $K_\mathrm{m}(z) = K_0$ and with the complex variables

$$\xi(z) = \eta(z) - \eta_\mathrm{g} \tag{69}$$

$$\eta(z) = u(z) + iv(z) \tag{70}$$

$$\eta_\mathrm{g} = u_\mathrm{g} + iv_\mathrm{g} \tag{71}$$

the equations of motion can be written in combined form as (the apostrophe denotes the partial derivative with respect to z)

$$\xi''(z) - i\frac{f_\mathrm{c}}{K_0}\xi(z) = 0 \tag{72}$$

The solution is

$$\xi(z) \propto \exp\left[-(1 + i)Az\right] \tag{73}$$

with $A = \sqrt{f_\mathrm{c}/2K_0}$ and $\xi(z \to \infty) = 0$.

The assumption K_m = const. is not a good one close to the ground where $K_\mathrm{m} \propto z$. It is therefore assumed that this solution is valid only above a certain height $z > h_1$ with a continuous and continuously differentiable connection at $z = h_1$ to some function $\xi_1(z) = \eta_1(z) - \eta_\mathrm{g}$ for $z \leq h_1$ with a given analytical profile $\eta_1(z)$. Thus

$$\xi(z > h_1) = \xi_0 \exp\left[-(1 + i)A(z - h_1)\right] \tag{74}$$

with the boundary conditions

$$\xi(h_1) = \xi_1(h_1) \tag{75}$$

$$\xi'(h_1) = \xi_1'(h_1) \tag{76}$$

or (η_g = const.)

$$\eta(h_1) = \eta_1(h_1) \tag{77}$$
$$\eta'(h_1) = \eta_1'(h_1) \tag{78}$$

The wind profile η_1 for $z \le h_1$ is parameterized as

$$\eta_1(z) = u_1(z) \exp\left[ia(z-h_a) + i\alpha_a\right] \tag{79}$$
$$\eta_1'(z) = \exp\left[ia(z-h_a) + i\alpha_a\right]\left[u_1'(z) + iau_1(z)\right] \tag{80}$$

The wind speed profile $u_1(z)$ is for example the usual log profile or a more elaborated one.

Note that in contrast to common analytical two-layer models,[3] a wind turn with height is accounted for also in the lowest layer. Such wind turn is observed in the atmosphere[4] and is also predicted by the numerical integration presented in the preceding section. Parameter a describes a constant gradient of the wind direction with a given direction α_a at height h_a.

Equation (74) and and boundary condition (77) yields

$$\eta(z) = \eta_g + \left[\eta_1(h_1) - \eta_g\right] \exp\left[-(i+1)A(z-h_1)\right] \tag{81}$$

The geostrophic wind vector η_g can be expressed using boundary condition (78) as

$$\eta_g = \eta_1(h_1) + (1-i)\frac{1}{2A}\eta_1'(h_1) \tag{82}$$

Inserting (82) into (81) and splitting into real and imaginary parts yields the final analytical expression:

[3]See for example: Zdunkowski, W., Bott, A. (2003): *Dynamics of the Atmosphere.* Cambridge University Press, Cambridge. Etling, D. (2008): *Theoretische Meteorologie.* Springer, Berlin.

[4]See for example: Brümmer, B., Schultze, M. (2015): *Analysis of a 7-year low-level temperature inversion data set measured at the 280m high Hamburg weather mast.* Meteorol. Z. **24**, 481-494. Janicke, U., Janicke, L. (2011): *Some aspects of the definition of meteorological boundary layer profiles and comparisons with measurements.* Reports on Environmental Physics Number 7, Edition 1, Janicke Consulting, ISSN 1439-8222 (German). http://www.janicke.de.

Lower layer $z \leq h_1$:

$$u(z) = u_1(z)\cos[\alpha_a + a(z - h_a)] \tag{83}$$
$$v(z) = u_1(z)\sin[\alpha_a + a(z - h_a)] \tag{84}$$

Upper layer $z > h_1$:

$$u(z) = u_1(h_1)c_1 + \frac{1}{2A}[(1 - c(z))\,p + s(z)q] \tag{85}$$
$$v(z) = u_1(h_1)s_1 + \frac{1}{2A}[(c(z) - 1)\,q + s(z)p] \tag{86}$$

with

$$c_1 = \cos[\alpha_a + a(h_1 - h_a)] \tag{87}$$
$$s_1 = \sin[\alpha_a + a(h_1 - h_a)] \tag{88}$$
$$p = u_1'(h_1)w_+ + a u_1(h_1)w_- \tag{89}$$
$$q = u_1'(h_1)w_- - a u_1(h_1)w_+ \tag{90}$$
$$w_+ = c_1 + s_1 \tag{91}$$
$$w_- = c_1 - s_1 \tag{92}$$
$$c(z) = \exp[-A(z - h_1)]\cos[A(z - h_1)] \tag{93}$$
$$s(z) = \exp[-A(z - h_1)]\sin[A(z - h_1)] \tag{94}$$
$$A = \sqrt{|f_c|/2K_0} \tag{95}$$

The values f_c, h_1, K_0, a, h_a, α_a, and the wind speed profile $u_1(z)$ must be explicitly provided.

Equations (85) and (86) apply to $f_c > 0$ (northern hemisphere). For $f_c < 0$ (southern hemisphere) they are replaced by

$$u(z) = u_1(h_1)c_1 + \frac{1}{2A}[(1 - c(z))\,q + s(z)p] \tag{96}$$
$$v(z) = u_1(h_1)s_1 - \frac{1}{2A}[(c(z) - 1)\,p + s(z)q] \tag{97}$$

3.2 Settings

A common parameterization of the wind speed profile in the lower layer is

$$u_1(z) = \frac{u_*}{\kappa} \begin{cases} \Psi_0\left(\frac{z}{L}\right) & \text{for} \quad 1/L \geq 0 \\ \left[\ln\left(\frac{z+z_0}{z_0}\right) - \Psi_1\left(\frac{z}{L}\right)\right] & \text{for} \quad 1/L < 0 \end{cases} \tag{98}$$

with

$$\Psi_0\left(\frac{z}{L}\right) = \ln\left(\frac{z+z_0}{z_0}\right) + 5\left(\frac{z}{L}\right) \tag{99}$$

$$\Psi_1\left(\frac{z}{L}\right) = \ln\left[\left(\frac{1+X}{1+X_0}\right)^2 \frac{1+X^2}{1+X_0^2}\right] - 2\left(\arctan X - \arctan X_0\right) \tag{100}$$

$$X = \left(1 - 15\frac{z+z_0}{L}\right)^{1/4} \quad , \quad X_0 = \left(1 - 15\frac{z_0}{L}\right)^{1/4} \tag{101}$$

This parameterization yields in combination with Eq. (64) for $K_\mathrm{m}(z)$ the useful relation

$$u_1'(z) = \frac{u_*^2}{K_\mathrm{m}(z; h_\mathrm{m} \to \infty)} \tag{102}$$

A reasonable choice for height h_1 is half the height at which $K_\mathrm{m}(z)$ takes its maximum value. With $K_\mathrm{m}(z)$ of Eq. (64) and applying for $1/L < 0$ the simpler analytical expression resulting from $1/L = 0$, the height h_1 can be expressed analytically as

$$h_1 \approx \begin{cases} \frac{L}{20}\left[\left(1 + \frac{10h_\mathrm{m}}{3\alpha L}\right)^{1/2} - 1\right] & \text{for} \quad 1/L \geq 0 \\ \frac{h_\mathrm{m}}{12\alpha} & \text{for} \quad 1/L < 0 \end{cases} \tag{103}$$

An estimate for K_0 is then

$$K_0 = K_\mathrm{m}(h_1) \tag{104}$$

The value of a can be estimated from the standard wind turn of $\pi/4$ across the Ekman layer of height π/A as

$$a = -0.2A \quad \text{(northern hemisphere)} \tag{105}$$

$$a = +0.2A \quad \text{(southern hemisphere)} \tag{106}$$

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3.3 Setting of the friction velocity

The analytical approximation requires a value for the friction velocity u_* that enters into the profile for the exchange coefficient.

If the friction velocity is not provided, its derivation is straightforward if a wind speed u_a at height $h_a \leq h_1$ is provided instead: the analytical wind profile $u_1(z; u_*)$ is proportional to u_* (Eq. 98) and thus $u_* = u_a/u_1(h_a; 1)$.

For $h_a > h_1$, which may occur at very stable stratification, u_* must be derived in an iterative procedure such that $\sqrt{u^2(h_a; u_*) + v^2(h_a; u_*)} = u_a$. The iterative procedure is also required for the numerical integration if $u_a(h_a)$ is provided instead of u_*.

Appendix

A Example profiles and comparisons

The following figures show some example profiles for different atmospheric stratifications (from very stable to very unstable).

In each figure, the red line denotes the profile of the exchange coefficient, the blue lines the profiles of wind speed, and the green lines the profiles of wind direction. Solid lines denote the result of the numerical integration, dashed lines the result of the analytical approximation. The gray circle on the red line denotes the position of the mixing layer height, the red circle the height h_1 of the first layer in the analytical approximation. The line in light blue denotes the wind speed profile of a simpler analytical approach ($u_1(z)$ of the analyical approximation for the first layer with correction terms for stable stratification).

The quality of the numerical solution with respect to measured profiles is mainly determined by the quality of the applied profile of the exchange coefficient. The quality of the analytical solution can be addressed by a comparison with the numerical solution, using the same assumption for the exchange coefficient.

The comparisons presented here as well as other comparisons show that the analytical approximation is usually quite close to the exact numerical solution. In general it slightly underestimates the overshooting of wind speed at stable stratification.

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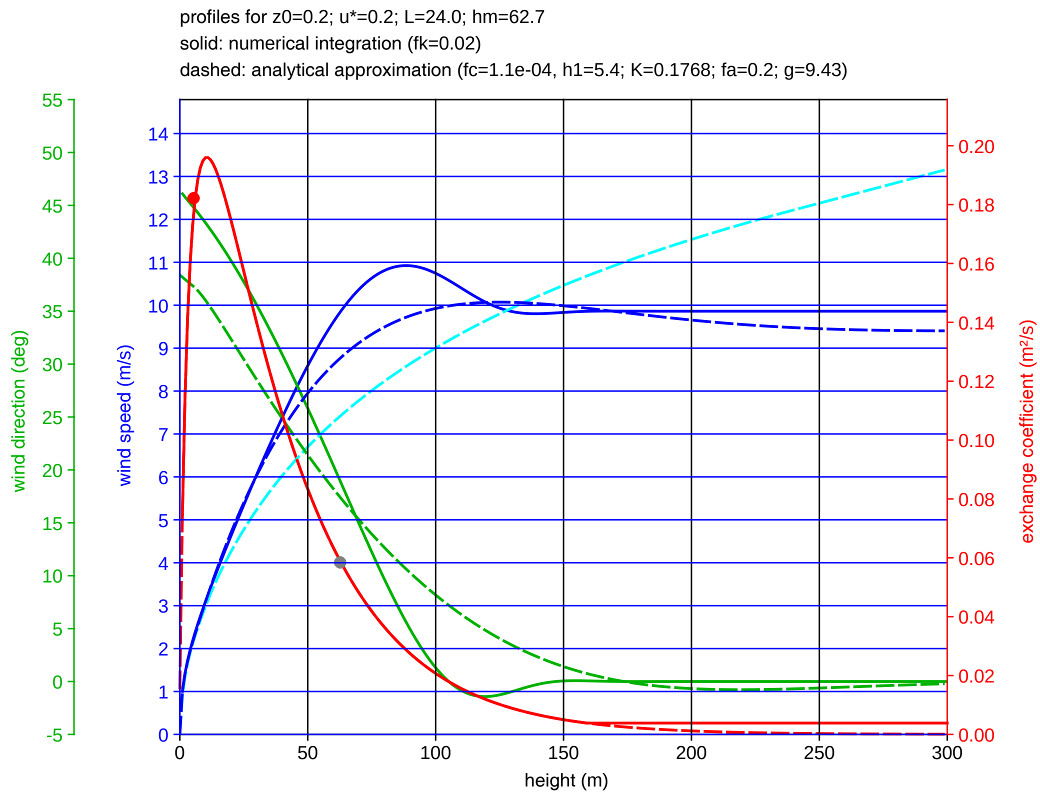
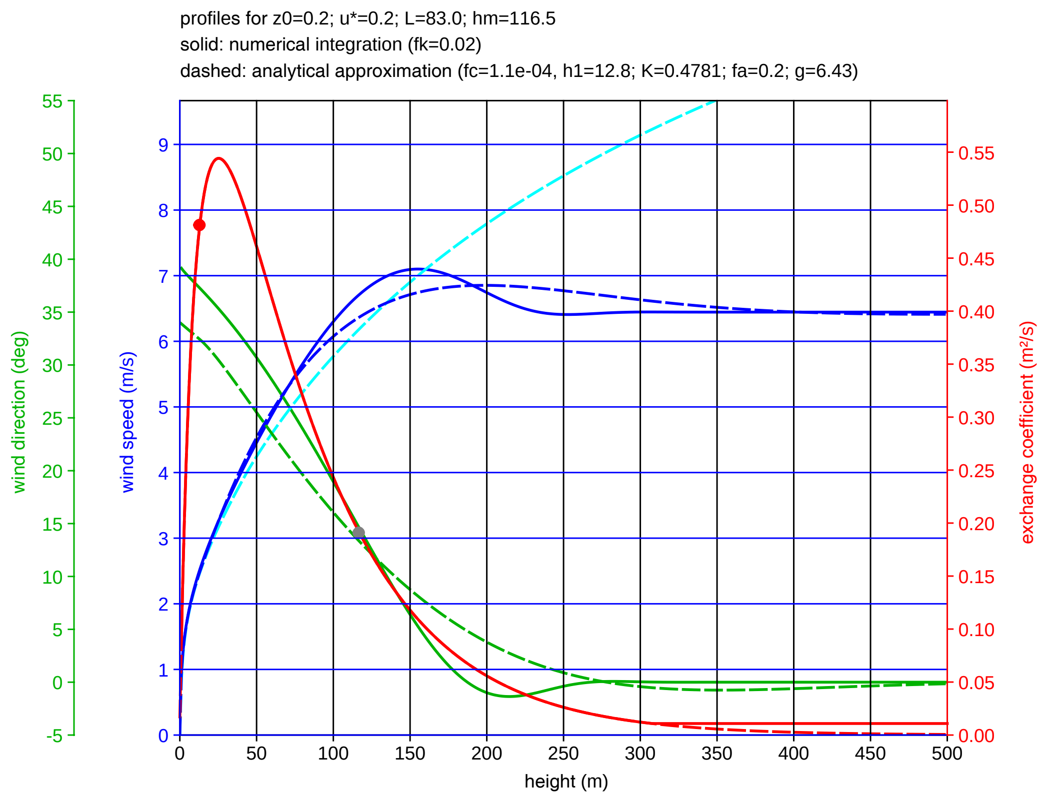
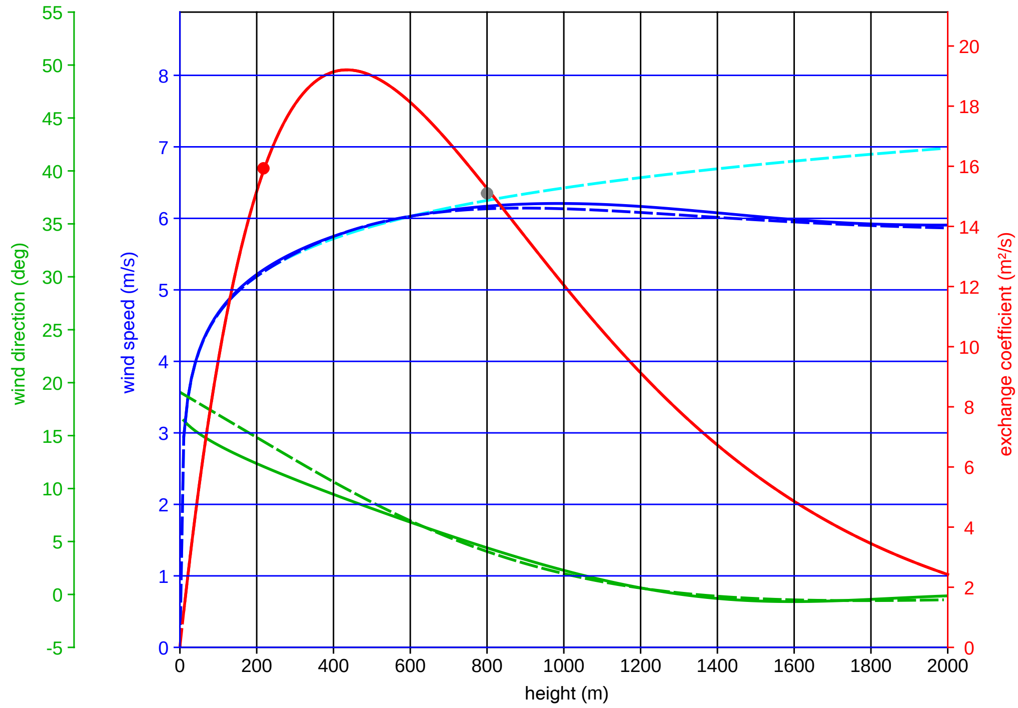


profiles for z0=0.2; u*=0.2; L=24.0; hm=62.7
solid: numerical integration (fk=0.02)
dashed: analytical approximation (fc=1.1e-04, h1=5.4; K=0.1768; fa=0.2; g=9.43)
wind direction (deg)
wind speed (m/s)
exchange coefficient (m²/s)
height (m)

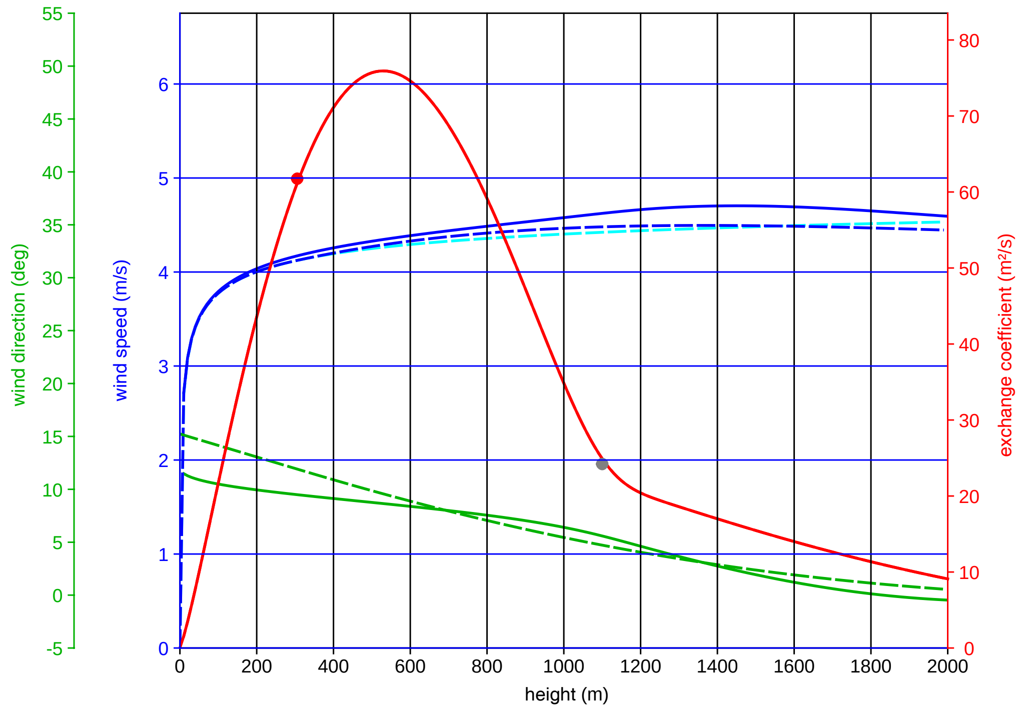


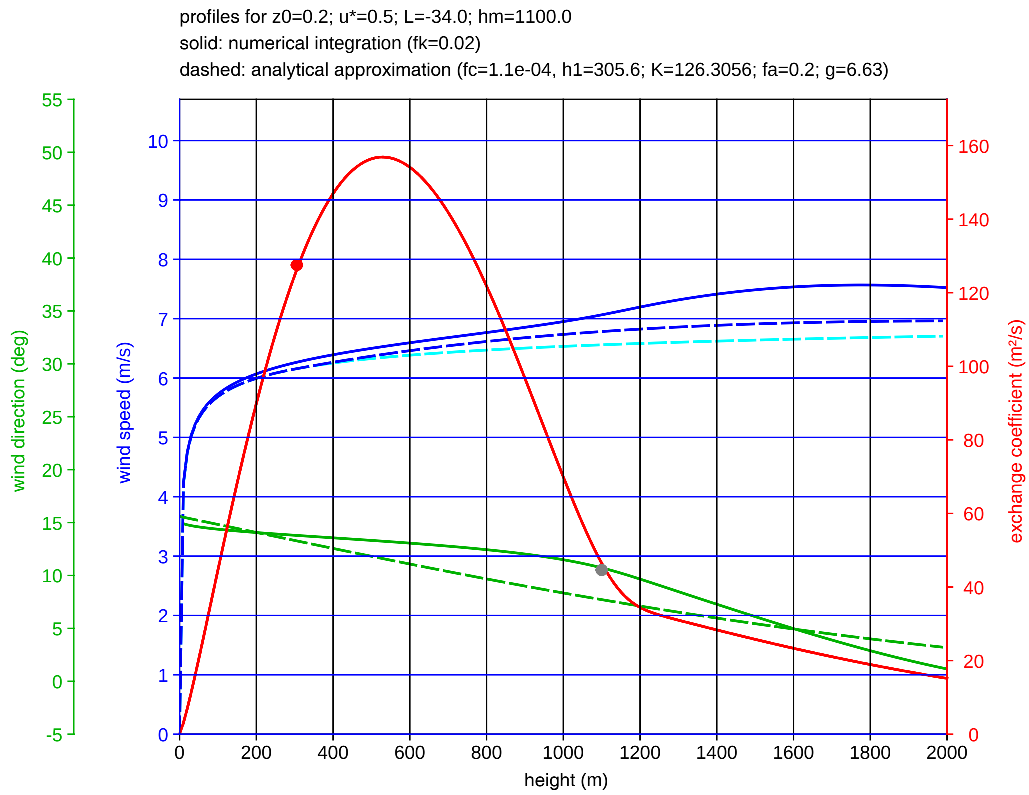
profiles for z0=0.2; u*=0.2; L=83.0; hm=116.5
solid: numerical integration (fk=0.02)
dashed: analytical approximation (fc=1.1e-04, h1=12.8; K=0.4781; fa=0.2; g=6.43)
wind direction (deg)
wind speed (m/s)
exchange coefficient (m²/s)
height (m)

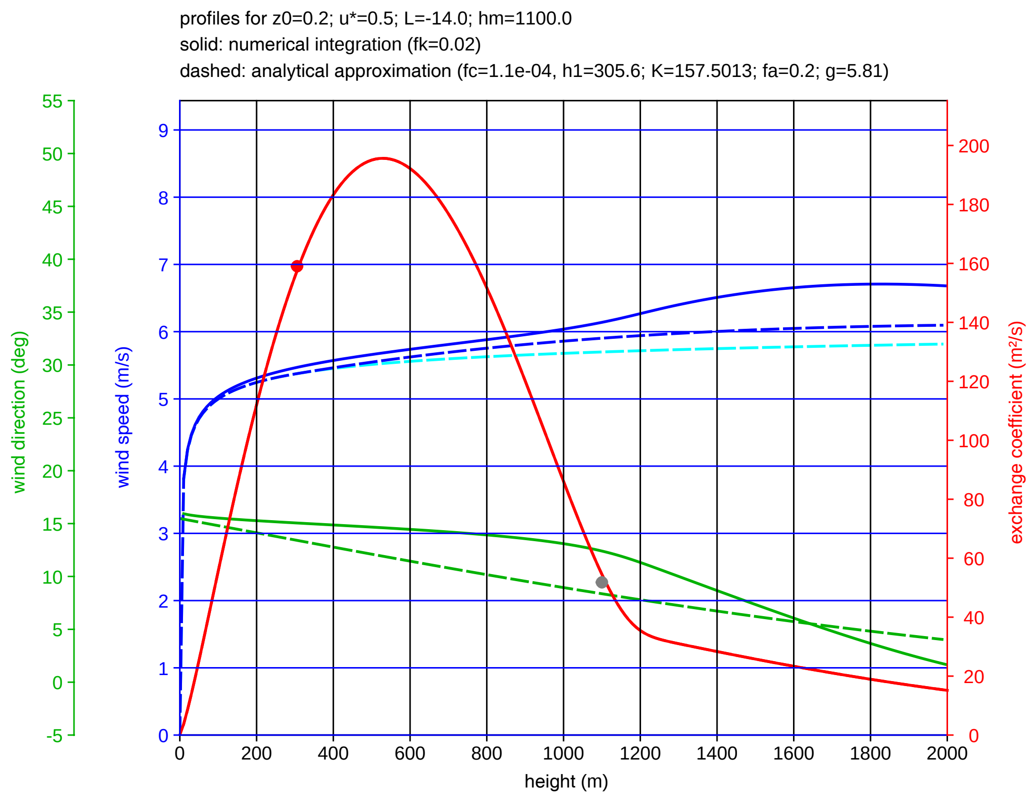
profiles for z0=0.2; u*=0.3; L=99999.0; hm=800.0
solid: numerical integration (fk=0.02)
dashed: analytical approximation (fc=1.1e-04, h1=217.5; K=15.8415; fa=0.2; g=5.85)



profiles for z0=0.2; u*=0.3; L=-81.0; hm=1100.0
solid: numerical integration (fk=0.02)
dashed: analytical approximation (fc=1.1e-04, h1=305.6; K=61.1551; fa=0.2; g=4.27)




profiles for z0=0.2; u*=0.5; L=-34.0; hm=1100.0
solid: numerical integration (fk=0.02)
dashed: analytical approximation (fc=1.1e-04, h1=305.6; K=126.3056; fa=0.2; g=6.63)
wind direction (deg)
wind speed (m/s)
exchange coefficient (m²/s)
height (m)


profiles for z0=0.2; u*=0.5; L=-14.0; hm=1100.0
solid: numerical integration (fk=0.02)
dashed: analytical approximation (fc=1.1e-04, h1=305.6; K=157.5013; fa=0.2; g=5.81)
wind direction (deg)
wind speed (m/s)
exchange coefficient (m²/s)
height (m)